Available online at www.sciencedirect.com
ScienceDirect
Journal of Geometry and Physics 57 (2007) 1999–2013
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

On the square of first order differential operators of Dirac type and the Einstein–Hilbert action

Jürgen Tolksdorf*

Max-Planck Institute for Mathematics in the Sciences, Mathematical Physics, Leipzig, Germany

Received 18 December 2006; received in revised form 19 March 2007; accepted 25 April 2007
Available online 29 April 2007

Abstract

The aim of this paper is to present a new global formula for the Lichnerowicz decomposition of the square of a general Dirac type first order differential operator. Concerning gauge theories, this formula permits re-writing the Yang–Mills action linearly in the curvature so that it becomes similar to the Einstein–Hilbert action. In fact, it is shown that the two action functionals can be expressed in identical geometric forms. This holds true even more also for Yang–Mills–Yukawa-like gauge theories. From this one infers that, in particular, the full bosonic action of the (minimal) Standard Model can be expressed geometrically in complete accordance with Einstein's theory of gravity.

MSC: 53C05; 53C07; 70S05; 70S15; 83C05

PACS: 02.40.Hw; 02.40.Ma; 04.20.-q; 14.80.Bn

Keywords: Dirac bundles; Dirac type differential operators; Linear connections; General relativity; Gauge theories; Standard Model

1. Motivation and the main statement

While the Yang–Mills action is quadratic, Einstein's theory of gravity is based on a functional which is *linear* in the curvature. This is known to have far reaching consequences. Over the last few decades many attempts have been made to recast Einstein's theory of gravity into a "Yang–Mills-like" form. In the following we reverse this point of view and show how Yang–Mills gauge theories can be expressed linearly in the curvature, recasting gauge theories into a "gravity-like" form. For this we discuss a canonical action functional within the geometrical setting of Dirac bundles in the light of a new global Lichnerowicz-like formula introduced in this article.

The paper is organized as follows. In the first section we present some motivation and the main statement of the paper. We also discuss some geometrically motivated restrictions on the manifold of all Dirac type first order differential operators. In the second section we present a proof of the main statement. This proof permits to gain some insight into the relation between Dirac type first order differential operators and connections on general Dirac

* Tel.: +49 (0)341 9959 733; fax: +49 (0)341 9959 658.
E-mail address: juergen.tolksdorf@mis.mpg.de.

0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2007.04.004

bundles. In particular, the proof shows that every Dirac type first order operator naturally induces a connection and thus a curvature on a the Dirac bundle. This is used in the third section which is devoted to the relation between the Yang–Mills and the Einstein–Hilbert action. This discussion is based on the results presented in the preceding sections which allow to express the canonical action, presented in the first section, in a purely geometrical form. The fourth section closes the article with the conclusion of the results presented, some general remarks and a brief outlook.

1.1. Motivation

We call in mind that a first order differential operator D is called of "Dirac type" provided its square satisfies

$$D^2 = -\triangle + V. \tag{1}$$

Here, $\triangle := -\mathrm{ev}_g(\nabla \circ \nabla) \overset{loc.}{=} -g^{ij}\left(\nabla_i \nabla_j - \Gamma^k_{ij}\nabla_k\right)$ denotes the *Bochner–Laplacian* defined by a connection ∇ on the Clifford module ("Dirac") bundle $\xi_{\mathrm{D}} \equiv (\mathcal{E}, M, \pi_{\mathcal{E}}, \gamma)$, and $V \in \Gamma(\mathrm{End}(\xi_{\mathrm{D}}))$ is a *zero order* differential operator. Both operators (∇, V) are uniquely determined by D. In the above local formula for the Bochner–Laplacian we used Einstein's summation convention with respect to a local chart $M \supset U \xrightarrow{x^k} \mathbb{R}$; Γ^k_{ij} are the corresponding Christoffel symbols with respect to a metric $g^{ij} \equiv g_{\mathrm{M}}(dx^i, dx^j)$ on M.

In the following we shall lay the basis for a mathematical discussion of the canonical functional

$$\begin{aligned} \mathcal{S}_{\mathrm{D}} &: \mathcal{D}(\xi_{\mathrm{D}}) \longrightarrow \mathbb{C} \\ D &\mapsto \int_{\mathrm{M}} *\mathrm{tr}_{\mathcal{E}} V \end{aligned} \tag{2}$$

on the manifold $\mathcal{D}(\xi_{\mathrm{D}})$ of all Dirac type first order differential operators compatible with the Clifford action γ.

It turns out that the *"universal Dirac action"* (2) is invariant with respect to the action of the diffeomorphism group $\mathrm{Diff}(\xi_{\mathrm{D}})$ of ξ_{D}. Accordingly, it is of interest to investigate the geometrical structure of the quotient set $\mathfrak{M}_{\mathrm{D}} := \mathcal{D}(\xi_{\mathrm{D}})/\mathrm{Diff}(\xi_{\mathrm{D}})$ and to analyze in what sense the Dirac functional (2) descends to this "moduli space of Dirac type operators". The main proposition of this article will provide us with a purely geometrical form of the *integrand* of the canonical functional (2). This in turn may help classify Dirac type differential operators on general Clifford module bundles. For this we present some natural restrictions on $\mathcal{D}(\xi_{\mathrm{D}})$. As mentioned already, we show in the third section that the universal Dirac action (2) actually provides a natural generalization of the action functional of general relativity.

The decomposition of D^2 into a second and zero order differential operator is known as *"general Lichnerowicz decomposition"* (see, for instance, in [5,7,3]). In particular, if $D := \gamma(\partial_{\mathrm{A}}) \equiv \not\partial_{\mathrm{A}}$ is defined in terms of a *Clifford connection* on a Clifford module bundle $\mathcal{E} \xrightarrow{\pi_{\mathcal{E}}} M$, i.e. a connection that satisfies the relation

$$[\partial_{\mathrm{A}}, \gamma(a)] = \gamma(\nabla^{\mathrm{Cl}} a) \tag{3}$$

for all sections a into the Clifford bundle of g_{M}, the relation (1) reads (cf. [15]):

$$\not\partial_{\mathrm{A}}^2 = \mathrm{ev}_g(\partial_{\mathrm{A}} \circ \partial_{\mathrm{A}}) + \gamma(\mathrm{F}_{\mathrm{A}}). \tag{4}$$

Here, respectively, ∇^{Cl} is the connection on the Clifford bundle that is induced by the Levi-Civitá connection of g_{M}, $\mathrm{F}_{\mathrm{A}} \in \Omega^2(M, \mathrm{End}(\mathcal{E}))$ is the curvature with respect to the Clifford connection ∂_{A}, and γ is the Clifford action of the cotangent bundle τ^*_{M} of M on ξ_{D}, whereby explicitly

$$\gamma(\mathrm{F}_{\mathrm{A}}) = \frac{1}{4} r_{\mathrm{M}} \mathrm{id}_{\mathcal{E}} + \gamma(F^{\mathcal{E}/S}_{\mathrm{A}}). \tag{5}$$

The smooth function r_{M} on M denotes the scalar curvature with respect to the metric g_{M}, and $F^{\mathcal{E}/S}_{\mathrm{A}} := F_{\mathrm{A}} - \not R$ is the "relative" ("twisting") curvature of ∂_{A}; $\not R$ is the curvature tensor with respect to ∇^{Cl} represented on the Clifford module bundle.

Consequently, the zero order operator V for Dirac type operators defined by Clifford connections can be expressed entirely in terms of the curvature of the Clifford connection that also determines the Bochner–Laplacian.

In this article we prove the following generalization of the well-known Lichnerowicz decomposition (4) of $\not\partial_{\mathrm{A}}^2$:

Main statement 1.1. *Let $\xi = (\mathcal{E}, M, \pi_{\mathcal{E}})$ be a complex vector bundle of finite rank over a (semi-)Riemannian manifold (M, g_{M}) of dimension $2n = p + q$ and arbitrary signature $s = p - q$. Let $\tau_{\mathrm{M}}^* \xrightarrow{\gamma} \mathrm{End}(\xi)$ be a Clifford mapping with respect to g_{M} which turns ξ into a Dirac bundle ξ_{D}. For every Dirac type first order differential operator*

$$D : \Gamma(\xi_{\mathrm{D}}) \longrightarrow \Gamma(\xi_{\mathrm{D}}) \tag{6}$$

there exists a connection ∂_{D} on ξ_{D} that is uniquely determined by D, so that

$$D^2 = \mathrm{ev}_{\mathrm{g}}(\partial_{\mathrm{D}} \circ \partial_{\mathrm{D}}) + \gamma(\mathrm{F}_{\mathrm{D}} + \partial_{\mathrm{D}}\omega_{\mathrm{D}} + \omega_{\mathrm{D}} \wedge \omega_{\mathrm{D}}). \tag{7}$$

Here, $\mathrm{F}_{\mathrm{D}} \in \Omega^2(M, \mathrm{End}(\mathcal{E}))$ is the curvature with respect to the connection ∂_{D}, and

$$\omega_{\mathrm{D}} := \Theta \wedge (D - \not\partial_{\mathrm{D}}) \in \Omega^1(M, \mathrm{End}(\mathcal{E})) \tag{8}$$

is the "Dirac form" associated with $D \in \mathcal{D}(\xi_{\mathrm{D}})$.

Moreover, the "Dirac connection"

$$\nabla_{\mathrm{D}} := \partial_{\mathrm{D}} + \omega_{\mathrm{D}} \tag{9}$$

is a canonical representative of D, i.e. $\gamma(\nabla_{\mathrm{D}}) = D$.

The canonical 1-form $\Theta \in \Omega^1(M, \mathrm{End}(\mathcal{E}))$ on ξ_{D} is basically given by the soldering form of the frame bundle of M lifted to the Clifford bundle τ_{Cl} with respect to g_{M} (for details, please see Section 2, Formula 6 in [22]). This canonical 1-form has the basic feature of being covariantly constant with respect to every Clifford connection ∂_{A}, i.e. $\partial_{\mathrm{A}}\Theta \equiv 0$.

We call the connection ∂_{D}, uniquely determined by D, the *"Bochner connection"* and (7) the *"general Lichnerowicz formula"*. Indeed, the formula (7) shares the two basic features of the Lichnerowicz formula (4): Each summand is fully determined by D itself and the zero order term is determined by the same connection that also determines the second order part in the decomposition (1). We shall see that the formula (7) will reduce to the Lichnerowicz formula (4) in the case where D is determined by a Clifford connection, i.e. $D = \not\partial_{\mathrm{A}}$. In this sense, the formula (7) may be regarded as a generalization of (4). Note, however, that the Bochner connection ∂_{D}, in general, does not represent D. That is, $\gamma(\partial_{\mathrm{D}}) \neq D$.

General remark concerning $\mathcal{D}(\xi_{\mathrm{D}})$: There are (at least) two obvious questions related to the domain of definition of the Dirac functional $\mathcal{S}_{\mathrm{D}}$:

1. The functional (2) is considered to be defined on (an appropriate sub-set of) the affine manifold of all Dirac type first order differential operators on a given Clifford module bundle $\xi_{\mathrm{D}} = (\xi, \gamma)$. Here, $\xi \equiv (\mathcal{E}, M, \pi_{\mathcal{E}})$ is a smooth (hermitian) vector bundle over a smooth (simply) connected, orientable (semi-)Riemannian manifold (M, g_{M}) of even dimension and γ denotes some given (reducible) Clifford action of the Clifford bundle τ_{Cl} associated with (M, g_{M}). However, since $\mathcal{S}_{\mathrm{D}}$ is also regarded as a functional of g_{M} the Clifford module structure of ξ_{D} is not considered to be fixed but, instead, results from a solution of the corresponding variational problem. To avoid the problem of the dependence of $\mathcal{D}(\xi_{\mathrm{D}})$ on the Clifford action γ, one may restrict to "twisted Grassmann bundles",
$$\xi_{\mathrm{D}} \simeq \tau_{\Lambda\mathrm{M}}^{\mathbb{C}} \otimes_{\mathrm{M}} \xi_{\mathrm{E}}, \tag{10}$$
which carry a natural Clifford action uniquely defined in terms of g_{M} (please, see also the corresponding remarks in Section 3). Here, $\tau_{\Lambda M}$ denotes the Grassmann bundle, $\tau_{\Lambda M}^{\mathbb{C}}$ its complexification and ξ_{E} some appropriate (hermitian) vector bundle.
2. In this paper we shall assume that either M is compact, or that for $D \in \mathcal{D}(\xi_{\mathrm{D}})$ the *"Dirac potential"*
$$V_{\mathrm{D}}(D) := \mathrm{tr}_{\mathcal{E}} V \in \mathcal{C}^\infty(M) \tag{11}$$
has an appropriate asymptotic behavior so that (2) is well defined. From a physical perspective either assumption seems spurious for a general space–time M. However, the following discussion is mainly concerned with the

integrand of (2) which is independent of the signature of the metric g_M and the "asymptotic behavior" of D. Hence, without loss of generality we can equally well restrict our discussion to the *universal Dirac Lagrangian*

$$\begin{aligned} \mathcal{L}_D : \mathcal{D}(\xi_D) &\longrightarrow \Omega^{\dim(M)}(M) \\ D &\mapsto *\mathrm{tr}_{\mathcal{E}} V, \end{aligned} \tag{12}$$

which defines the Dirac functional $\mathcal{S}_D$ instead of the functional itself.

Whenever the latter is considered, however, either of the above mentioned assumptions is (implicitly) assumed to hold true. This, however, will give rise to analytical restrictions on the domain of definition of $\mathcal{S}_D$, which are not discussed in what follows.

1.2. Some geometrical constrains on $\mathcal{D}(\xi_D)$

In the form (7) the general Lichnerowicz decomposition (1) may be helpful for classifying Dirac type first order differential operators. This in turn may provide some "regularity conditions" on $\mathfrak{M}_D$ analogous to the (anti-)self-duality condition in the case of pure Yang–Mills theory (please, see also our discussion at the end of the paper).

If, for instance, we impose on D the condition to solve the differential equation

$$\partial_D \omega_D + \omega_D \wedge \omega_D \doteq 0, \tag{13}$$

which is equivalent to restricting D to those $D \in \mathcal{D}(\xi_D)$ for which

$$D^2 = \mathrm{ev}_g(\partial_D \circ \partial_D) + \gamma(\mathrm{F}_D), \tag{14}$$

we find that, in this form the general Lichnerowicz formula (7) is fully analogous to the original Lichnerowicz formula (4). This holds true especially for the specific sub-class of solutions of (13) that is given by those $D \in \mathcal{D}(\xi_D)$ also satisfying

$$\nabla_D \doteq \partial_D. \tag{15}$$

The unique solution is then provided by Dirac type operators that are defined in terms of Clifford connections, i.e.

$$\nabla_D = \partial_D = \partial_A. \tag{16}$$

Note that each connection class contains exactly one Dirac connection. This is because ∂_D is uniquely defined by D (please, see the injection (27) below). Consequently, every connection class has at most one Clifford connection.

From (5) it follows that for $D = \partial\!\!\!/_A$, the universal Dirac functional becomes proportional to the Einstein–Hilbert functional of general relativity. In particular, for $D = \partial\!\!\!/_A$ the functional (2) is independent of the twisted part of the Clifford connection (please, see also our discussion in Section 3).

Another natural condition imposed on D is provided by the weaker requirement that only the Bochner connection coincides with a Clifford connection, i.e.

$$\partial_D \doteq \partial_A. \tag{17}$$

In this case, the Dirac forms turn out to have a very particular form and the unique solution of (17) is given by the Dirac connections

$$\nabla_D = \partial_A + \Theta \wedge (\gamma_M \otimes \phi), \tag{18}$$

which are known to play a fundamental role in mathematics as well as in physics (cf. [18,4,6,21]). In particular, Dirac type operators satisfying the condition (17) are very much related to *spontaneous symmetry breaking* in gauge theories. In fact, under the condition (17), the critical points of the Dirac functional (2) turn out to spontaneously break the underlying gauge symmetry if $\phi \neq 0$. This is known to play a basic role in the geometrical description of the standard model of elementary particles (cf. Proposition 3.1 in [22]). In (18), respectively, $\gamma_M \in \Gamma(\tau_{Cl})$ is proportional to the volume form $\mu_M \in \Omega^{2n}(M)$ defined by g_M, and $\phi \in \Omega^0(M, \mathrm{End}(\mathcal{E}))$ is a section which super-commutes with the Clifford action γ. However, the universal Dirac action is still independent of the twisting part of ∂_A even for these slightly more general Dirac type operators uniquely defined by the condition (17).

Yet another geometrically natural restriction of the Dirac functional (2) is given by those $D \in \mathcal{D}(\xi_D)$ satisfying

$$\mathrm{div}_D\omega_D := \mathrm{ev}_g\,(\partial_D\omega_D) \doteq 0. \tag{19}$$

Hence, the zero order operator associated with D is given by the ("quantized") curvature of the Dirac connection $\nabla_D \in \mathcal{A}(\xi_D)$:

$$V = \gamma(\mathrm{F}_{\nabla_D}). \tag{20}$$

In the third section we shall come back to this particular form of the endomorphism $V \in \Gamma(\mathrm{End}(\xi_D))$ that is naturally associated with $D \in \mathcal{D}(\xi_D)$. Indeed, we shall prove that for any Dirac type first order differential operator $D \in \mathcal{D}(\xi_D)$ the two top forms $*\mathrm{tr}_{\mathcal{E}}V$ and $*\mathrm{tr}_{\mathcal{E}}(\gamma(\mathrm{F}_{\nabla_D}))$ on M define the same cohomology class in $\mathrm{H}^{2n}_{dR}(M)$. This will also provide us with a simple geometrical interpretation of the condition (19).

2. The proof of the main statement

In this section formula (7) is proved. The way the proof is presented yields some insight into the relation between connections and Dirac type first order operators on a general Clifford module bundle. It is well known that there is a one-to-one correspondence between Dirac type operators and "Clifford super connections" (e.g. see [3]). Indeed, every Dirac type operator only corresponds to an equivalence class of connections on a Clifford module bundle. However, in the following we show that every such equivalence class has a natural representative, called the "Dirac connection" that is induced by the Dirac type operator considered. In this section we also discuss this fact from a global geometrical point of view. We show that, in contrast to the Bochner connection uniquely determined by a Dirac type operator, the Dirac connection can be always geometrically interpreted as global section of a certain principal fibering. It is the curvature of these distinguished sections (which allow to make the decomposition of the square of any Dirac type operator most similar to the usual Lichnerowicz formula of Dirac type operators defined in terms of a Clifford connections) that yields a natural generalization of the Einstein–Hilbert Lagrangian density as will be discussed in some detail in Section 3.

To prove the statement (1.1) we first summarize some facts on Dirac type first order differential operators.

2.1. Preliminaries

Let again $\xi_D := (\mathcal{E}, M, \pi_{\mathcal{E}}, \gamma)$ be a Clifford module bundle of finite rank over a smooth, orientable, (simply-) connected (semi-)Riemannian manifold (M, g_M) of dimension $2n = p + q$ and arbitrary signature $s = p - q$.

The Clifford action $\tau^*_M \times_M \xi_D \xrightarrow{\gamma} \xi_D$ yields the mapping

$$\begin{aligned}&\delta_\gamma : \Omega^1(M, \mathrm{End}(\mathcal{E})) \longrightarrow \Omega^0(M, \mathrm{End}(\mathcal{E}))\\ &\alpha \mapsto \gamma(\alpha)\end{aligned} \tag{21}$$

which has a canonical right inverse

$$\begin{aligned}&\mathrm{ext}_\Theta : \Omega^0(M, \mathrm{End}(\mathcal{E})) \longrightarrow \Omega^1(M, \mathrm{End}(\mathcal{E}))\\ &\Phi \mapsto \Theta \wedge \Phi.\end{aligned} \tag{22}$$

Let, respectively, $\mathcal{A}(\xi_D)$ be the affine set of all (linear) connections on ξ_D and $\mathcal{D}(\xi_D)$ be the affine set of all Dirac type operators on ξ_D compatible with the Clifford action γ. We denote by $\wp := \mathrm{ext}_\Theta \circ \delta_\gamma$ the corresponding idempotent associated with the Clifford action γ. Then,

$$\mathcal{D}(\xi_D) \simeq \mathcal{A}(\xi_D)/\ker(\delta_\gamma). \tag{23}$$

Consequently, there is a one-to-one correspondence between Dirac type operators and equivalence classes of connections on ξ_D. However, it turns out that each connection class has a natural representative.

2.2. The geometrical meaning of the Bochner and the Dirac connection

We make use of the following map

$$\begin{aligned}\Xi : \mathcal{A}(\xi_{\mathrm{D}}) &\longrightarrow \Omega^1(M, \mathrm{End}(\mathcal{E}))\\ \nabla &\mapsto \Xi_\nabla,\end{aligned} \tag{24}$$

where, locally,

$$\Xi_\nabla \overset{\text{loc.}}{=} -\frac{1}{2} g_{li} X^l \otimes \gamma(X^j)\left([\nabla_{X_j}, \gamma(X^i)] + \omega^i_{jk}\gamma(X^k)\right) \tag{25}$$

with $(X^1, \dots, X^n)$ being a local co-frame on M and $(X_1, \dots, X_n)$ its dual. The symbols $\omega^i_{jk} := X^i(\nabla^{\mathrm{TM}}_{X_j} X_k)$ are the corresponding Levi-Civitá connection coefficients with respect to g_{M} and the chosen frame and $g_{ij} \equiv (g_{\mathrm{M}}(X^i, X^j))^{-1}$.

One could consider Ξ as "measuring" how much ∇ deviates from being a Clifford connection. The form Ξ has been introduced already in [1]. We then have the following

Lemma 2.1. *The affine mapping*

$$\begin{aligned}\Pi : \mathcal{A}(\xi_{\mathrm{D}}) &\longrightarrow \mathcal{A}(\xi_{\mathrm{D}})\\ \nabla &\mapsto \nabla + \Xi_\nabla\end{aligned} \tag{26}$$

is well defined on $\mathcal{A}(\xi_{\mathrm{D}})/\ker(\delta_\gamma)$. *Consequently, one has a well-defined mapping*

$$\begin{aligned}\Pi_{\mathrm{D}} : \mathcal{D}(\xi_{\mathrm{D}}) &\longrightarrow \mathcal{A}(\xi_{\mathrm{D}})\\ D &\mapsto \partial_{\mathrm{D}} := \Pi(\nabla)\end{aligned} \tag{27}$$

with $\gamma(\nabla) = D$. *We call the elements of* $\mathrm{im}(\Pi_{\mathrm{D}}) \subset \mathcal{A}(\xi_{\mathrm{D}})$ *the "Bochner connections" on the Clifford module bundle* ξ_{D}.

Proof. Let $\nabla, \nabla' \in \mathcal{A}(\xi_{\mathrm{D}})$. Then, $\gamma(\nabla) = \gamma(\nabla')$ iff $\nabla' - \nabla \in \ker(\delta_\gamma) \subset \Omega^1(M, \mathrm{End}(\mathcal{E}))$. Moreover, $\Xi_{\nabla'} - \Xi_\nabla = \nabla - \nabla'$ and hence $\Pi(\nabla') = \Pi(\nabla)$. □

Note that $\ker(\delta_\gamma) \simeq \ker(\Pi_{\mathrm{D}})$ and thus

$$\mathcal{D}(\xi_{\mathrm{D}}) \simeq \mathrm{im}(\Pi_{\mathrm{D}}). \tag{28}$$

This equivalence gives the geometrical meaning of the differential form Ξ_∇ associated with a connection $\nabla \in \mathcal{A}(\xi_{\mathrm{D}})$ on a Clifford module bundle ξ_{D}. Note that in general $\gamma(\partial_{\mathrm{D}}) \neq D$. However, every $D \in \mathcal{D}(\xi_{\mathrm{D}})$ has a natural representative within the quotient $\mathcal{A}(\xi_{\mathrm{D}})/\ker(\delta_\gamma)$.

Indeed, there is a natural lift

$$\begin{aligned}\mathcal{A}(\xi_{\mathrm{D}})/\ker(\delta_\gamma) &\longrightarrow \mathcal{A}(\xi_{\mathrm{D}})\\ [\nabla] &\mapsto \nabla_{\mathrm{D}} := \Pi(\nabla) + \wp(\nabla - \Pi(\nabla)).\end{aligned} \tag{29}$$

Note that each of the summands is uniquely defined by $D \in \mathcal{D}(\xi_{\mathrm{D}})$, which corresponds to $[\nabla] \in \mathcal{A}(\xi_{\mathrm{D}})/\ker(\delta_\gamma)$. We call (29) the *Dirac connection* on ξ_{D} that corresponds to D.

Lemma 2.2. *For* $D \in \mathcal{D}(\xi_{\mathrm{D}})$, *let* $\omega_{\mathrm{D}} := \mathrm{ext}_\Theta(D - \partial\!\!\!/_{\mathrm{D}}) \in \Omega^1(M, \mathrm{End}(\mathcal{E}))$ *be the associated Dirac form such that the Dirac connection reads*

$$\nabla_{\mathrm{D}} = \partial_{\mathrm{D}} + \omega_{\mathrm{D}}. \tag{30}$$

Like the Bochner connection the Dirac form only depends on D and

$$\Xi_{\nabla_{\mathrm{D}}} = -\omega_{\mathrm{D}}. \tag{31}$$

Proof. Let $\partial_{\rm D}$ be the Bochner connection with respect to D. Since $\omega_{\rm D} = \wp(\nabla - \Pi(\nabla))$ it is clear that the Dirac form is independent of $\ker(\delta_\gamma) \subset \Omega^1(M, \mathrm{End}(\mathcal{E}))$. Moreover, since $\gamma(\nabla_{\rm D}) = D$ we have

$$\begin{aligned}\partial_{\rm D} &= \Pi(\nabla_{\rm D})\\ &= \nabla_{\rm D} + \Xi_{\nabla_{\rm D}}\\ &= \partial_{\rm D} + \omega_{\rm D} + \Xi_{\nabla_{\rm D}},\end{aligned} \tag{32}$$

which proves the statement. □

To obtain a geometrical interpretation of the Dirac connections on $\xi_{\rm D}$, let $\mathcal{T}_\gamma := \ker(\wp)$ be the translational subgroup of $\Omega^1(M, \mathrm{End}(\mathcal{E})) \simeq \mathrm{T}_\nabla\mathcal{A}(\xi_{\rm D})$. The fibering

$$\begin{aligned}&\mathcal{T}_\gamma \hookrightarrow \mathcal{A}(\xi_{\rm D}) \twoheadrightarrow \mathcal{D}(\xi_{\rm D})\\ &\nabla \mapsto D\end{aligned} \tag{33}$$

is a smooth principal $\mathcal{T}_\gamma$ bundle which admits a natural section

$$\begin{aligned}&\sigma_{\rm D} : \mathcal{D}(\xi_{\rm D}) \longrightarrow \mathcal{A}(\xi_{\rm D})\\ &D \mapsto \nabla_{\rm D}.\end{aligned} \tag{34}$$

We are thus allowed to make the identification:

$$\mathcal{A}(\xi_{\rm D}) = \mathcal{D}(\xi_{\rm D}) \times \wp^\perp(\Omega^1(M, \mathrm{End}(\mathcal{E}))), \tag{35}$$

where $\wp^\perp := \mathrm{id} - \wp$ is the complementary idempotent.

Although $\mathcal{T}_\gamma \simeq \ker(\delta_\gamma)$, the above identification is a slightly stronger statement than the isomorphism (23). Note that, in contrast to $\sigma_{\rm D}$, the mapping (27) does not define a section of the principal fibering (33).

2.3. *The zero order operator V*

We now turn to the proof of the first part of the statement (1.1). For this we make use of the following formula for the general Lichnerowicz decomposition (1):

$$\triangle = -\mathrm{ev}_g(\partial_{\rm D} \circ \partial_{\rm D}), \tag{36}$$

$$V = \gamma(\mathrm{F}_\nabla) + \mathrm{ev}_g\left(\nabla\Xi_\nabla + \Xi^2_\nabla\right) \tag{37}$$

with $\nabla \in \mathcal{A}(\xi_{\rm D})$ being any representative of D. The validity of these formulas was proved in [1] in the case of a positive signature. In particular, it has been shown that they are independent of the choice of a Clifford connection. In [21] it was shown that these formulas are actually independent of any representative ∇ (finally, [22] treats the case of arbitrary signature). The main drawback of the formula (37) is that, in contrast to the Bochner–Laplacian, it is defined in terms of the arbitrary choice of a representative $\nabla \in \mathcal{A}(\xi_{\rm D})$ of D. It is thus not quite analogous to (4). Moreover, each summand in (37) strongly depends on the choice of ∇. Only the sum is shown to be independent of this choice.

Theorem 2.1. *Let $D \in \mathcal{D}(\xi_{\rm D})$ be a generalized Dirac operator. The zero order part V of D^2 can be expressed in terms of the Bochner connection uniquely associated with D:*

$$V = \gamma(\mathrm{F}_{\rm D} + \partial_{\rm D}\omega_{\rm D} + \omega_{\rm D} \wedge \omega_{\rm D}), \tag{38}$$

where $\mathrm{F}_{\rm D} \in \Omega^2(M, \mathrm{End}(\mathcal{E}))$ is the curvature on $\xi_{\rm D}$ with respect to the Bochner connection $\partial_{\rm D} \in \mathcal{A}(\xi_{\rm D})$ and $\omega_{\rm D} := \mathrm{ext}_\Theta(D - \not{\partial}_{\rm D})$. Here, both $\mathrm{F}_{\rm D}$ and $\partial_{\rm D}\omega_{\rm D} + \omega_{\rm D} \wedge \omega_{\rm D}$ are considered as sections of the bundle $\tau_{\rm Cl} \otimes_{\rm M} \mathrm{End}(\xi_{\rm D})$.

Expressed in this form, the zero order part of D^2 formally looks like the "quantized" curvature of the *Dirac connection* $\nabla_{\rm D}$ on $\xi_{\rm D}$. However, the section

$$\partial_{\rm D}\omega_{\rm D} \in \Gamma(\tau_{\rm Cl} \otimes_{\rm M} \mathrm{End}(\xi_{\rm D})) \tag{39}$$

also contains a "symmetric" part within $\tau_{\rm Cl}$.

Proof. Since the zero order operator V is independent of the connection $\nabla \in \mathcal{A}(\xi_D)$ representing D, we may replace ∇ in (37) by the Dirac connection (30) to obtain

$$V = \gamma(\mathrm{F}_{\nabla_D}) + \mathrm{ev}_g\left(\nabla_D \Xi_{\nabla_D} + \Xi^2_{\nabla_D}\right). \tag{40}$$

We use Lemma 2.2 and the specific form of the Dirac connection to again re-write (37) as

$$V = \gamma(\mathrm{F}_D) + \gamma(\omega_D \wedge \omega_D) + \gamma(\mathrm{d}_D\omega_D) \pm \mathrm{ev}_g(\partial_D\omega_D). \tag{41}$$

Here, $\mathrm{F}_D \in \Omega^2(M, \mathrm{End}(\mathcal{E}))$ is the curvature on ξ_D with respect to the Bochner connection $\partial_D \in \mathcal{A}(\xi_D)$, and d_D denotes the induced exterior covariant derivative with respect to ∂_D. The formula (38) then follows from the well-known linear isomorphism between the Clifford and the Grassmann algebra, whereby the Clifford product (denoted by juxtaposition) can be expressed as

$$\alpha\beta = \pm g_M(\alpha, \beta) + \alpha \wedge \beta \tag{42}$$

for all $\alpha, \beta \in \mathrm{T}^*M \hookrightarrow Cl(M, g_M)$ (the latter is the total space of the Clifford bundle). □

To prove the validity of the formula for the Bochner–Laplacian (36) is simply a copy of the proof already presented in [1]. There it was also shown that the Bochner–Laplacian is independent of the choice of a Clifford connection. That the Bochner connection is indeed unique follows once again from the isomorphism (28). This finally proves the validity of the assertion (1.1).

In the next section we discuss the Yang–Mills functional and the Einstein–Hilbert action of gravity in terms of the generalized Lichnerowicz formula (7).

3. Gauge theories and the Einstein–Hilbert action

In contrast to the Yang–Mills functional, the Einstein–Hilbert functional is defined linearly in the curvature[1]:

$$\mathcal{S}_{EH}(g_M) := \kappa_{EH} \int_M *r_M. \tag{43}$$

Here, $g_M \in \Gamma(\xi_{EH})$ is considered as being a section of the "Einstein–Hilbert bundle" ξ_{EH} :

$$\pi_{EH} : E_{EH} \equiv FM \times_{\mathrm{GL(2n)}} \mathrm{GL(2n)/SO(p,q)} \longrightarrow M, \tag{44}$$

where FM is the total space of the frame bundle of M and "$*$" is the Hodge map with respect to such a section and a chosen orientation of M. Again, the function $r_M \in \mathcal{C}^\infty(M)$ denotes the scalar curvature of the base manifold M with respect to the (metric equivalent) section $g_M \in \Gamma(\xi_{EH})$. However, the gravity action functional may be expressed also in terms of a curvature on the total space of a Dirac bundle.

Let $\xi_D := \tau_{\Lambda M}$ be the Grassmann bundle of M (with total space ΛM) and $D := d + \delta$ the "Gauss–Bonnet" ("Hodge–de Rham") operator acting on $\Gamma(\xi_D) = \Omega(M) = \Omega^+(M) \oplus \Omega^-(M)$. Since $\tau_{Cl} \overset{\text{lin.}}{\simeq} \tau_{\Lambda M}$ there exists a natural Clifford action $\tau^*_M \xrightarrow{\gamma} \mathrm{End}(\tau_{\Lambda M})$ on the Grassmann bundle $\tau_{\Lambda M}$, turning the latter into a Dirac bundle so that

$$D = \gamma(\partial) \equiv \not\partial. \tag{45}$$

Here, ∂ is the (canonical) Clifford connection on $\tau_{\Lambda M}$ that is induced by the (semi-)Riemannian connection with respect to g_M. The Einstein–Hilbert action (43) may be re-written as

$$\mathcal{S}_{EH}(g_M) = \tilde{\kappa}_{EH} \int_M *\mathrm{tr}_{\Lambda M}\gamma(\mathcal{R}), \tag{46}$$

where $\mathcal{R} \in \Omega^2(M, \mathrm{End}(\Lambda \mathrm{M}))$ is the curvature on the total space of the specific Dirac bundle $\tau_{\Lambda M}$ with respect to ∂. Accordingly, $\gamma(\mathcal{R}) \in \Gamma(\mathrm{End}(\tau_{\Lambda M}))$, and $\mathrm{tr}_{\Lambda M}$ denotes the (fiber-wise) trace on the bundle $\mathrm{End}(\tau_{\Lambda M})$.

[1] For simplicity, we skip the (numerical) details of the constants adjusting the correct physical dimensions which depend on the choice of units.

There are two reasons for recasting the Einstein–Hilbert functional (43) into the form (46): First, it shows that the gravity action can be expressed also in terms of the curvature on (the total space of) a certain Dirac bundle. Second, in this way the Einstein–Hilbert functional can easily be generalized to arbitrary Dirac bundles $\xi_{\rm D}$. For this one replaces

$$\partial \rightsquigarrow \partial_{\rm A}, \tag{47}$$

so that the Einstein–Hilbert action reads:

$$\mathcal{S}_{\rm EH}(g_{\rm M}) = \lambda_{\rm EH} \int_{\rm M} *{\rm tr}_{\mathcal{E}}\gamma({\rm F_A}), \tag{48}$$

where again, $\mathrm{F_A} \in \Omega^2(M, \mathrm{End(E)})$ is the curvature on $\mathcal{E}$ with respect to the Clifford connection $\partial_{\rm A}$. Accordingly, $\gamma(\mathrm{F_A}) \in \Gamma(\mathrm{End}(\xi_{\rm D}))$, and ${\rm tr}_{\mathcal{E}}$ denotes the (fiber-wise) trace on the endomorphism bundle $\mathrm{End}(\xi_{\rm D})$.

As mentioned already in the first section, the Einstein–Hilbert action (48) is but the (appropriately normalized) Dirac functional (2) evaluated with respect to $\not\partial_{\rm A}$:

$$\mathcal{S}_{\rm EH}(g_{\rm M}) \sim \mathcal{S}_{\rm D}(\not\partial_{\rm A}). \tag{49}$$

In fact, this Dirac functional is independent of the (twisting part of the) chosen Clifford connection (47). The reason for this replacement is that, contrary to $\partial_{\rm A}$, the operator ∂ has no global meaning on a general Dirac bundle. Hence, by an arbitrary choice of a Clifford connection, the Einstein–Hilbert action can be written as a specific Dirac action on a general Dirac bundle. Of course, with regard to the Grassmann bundle, ∂ is the distinguished Clifford connection which defines the Gauss–Bonnet operator.

General remark on the role of the Clifford mappings γ:

For every Dirac bundle $\xi_{\rm D} = (\mathcal{E}, M, \pi_{\mathcal{E}}, \gamma)$ over an even dimensional, orientable (semi-)Riemannian manifold $(M, g_{\rm M})$ one has (see [2] and, for example, in [3])

$$\mathrm{End}(\xi_{\rm D}) \simeq \tau_{\rm Cl}^{\mathbb{C}} \otimes_{\rm M} \mathrm{End}_{\rm Cl}(\xi_{\rm D}). \tag{50}$$

Basically, this follows from the famous *Wedderburn theorems* (cf. Theorem 11.16.1 in [8]). Here, $\tau_{\rm Cl}^{\mathbb{C}}$ denotes the complexified Clifford bundle and $\mathrm{End}_{\rm Cl}(\xi_{\rm D})$ the sub-bundle of endomorphisms on $\xi_{\rm D}$ which super-commute with the Clifford action γ. As a consequence, the total space $\mathcal{E}$ of any Dirac bundle $\xi_{\rm D}$ locally looks like a "twisted spinor bundle", i.e. locally $\mathcal{E} \simeq S \otimes_{\rm U} W$ with $\iota : U \hookrightarrow M$ being a local (simply connected) sub-set of M and $S \twoheadrightarrow U$ the (local) spinor bundle with respect to $g_{\rm U} := \iota^* g_{\rm M}$ and the induced orientation of $U \subset M$. The corresponding (local) vector bundle $W \twoheadrightarrow U$ is given by $W := \mathrm{Hom}_{\rm Cl}(S, \mathcal{E})$. Hence, $\tau_{\rm Cl}^{\mathbb{C}} \overset{\rm loc.}{\simeq} \mathrm{End}(S)$ and $\mathrm{End}_{\rm Cl}(\mathcal{E}) \overset{\rm loc.}{\simeq} \mathrm{End}(W)$. In particular, if M is a spin manifold then

$$\xi_{\rm D} \simeq \tau_{\rm S} \otimes_{\rm M} \xi_{\rm W} \tag{51}$$

where $\tau_{\rm S}$ denotes the spinor bundle (with respect to a chosen spin structure Λ) and $\xi_{\rm W} \equiv (W, M, \pi_{\rm W})$.

Because of (50), a Clifford action

$$\tau_{\rm Cl} \times_{\rm M} \xi_{\rm D} \xrightarrow{\gamma} \xi_{\rm D} \tag{52}$$

is seen to correspond to the bundle $\mathrm{End}_{\rm Cl}(\xi_{\rm D})$. For example, if M admits a spin structure Λ, then for $\xi_{\rm D} := \tau_{\Lambda \rm M}^{\mathbb{C}}$ one has $\mathrm{End}_{\rm Cl}(\xi_{\rm D}) \simeq \mathrm{End}(S^*)$. This is because M is assumed to be even dimensional. Note that for M being spin there is a natural Clifford module structure on $\tau_{\Lambda \rm M}^{\mathbb{C}}$ that is fully determined by $(g_{\rm M}, \Lambda)$. This reduces further to $g_{\rm M}$ if M is also assumed to be simply connected. In fact, this can be generalized.

Although there is not a unique Clifford action even on Dirac bundles $\xi_{\rm D}$ fulfilling

$$\xi_{\rm D} \simeq \tau_{\Lambda \rm M}^{\mathbb{C}} \otimes_{\rm M} \xi_{\rm E}, \tag{53}$$

with $\xi_{\rm E} \equiv (E, M, \pi_{\rm E})$ being some given (hermitian) vector bundle, there is a natural Clifford action γ on any *"twisted Grassmann bundle"* (53) according to the canonical (linear) identification $\tau_{\Lambda \rm M} \simeq \tau_{\rm Cl}$. This particular Clifford action γ is then considered to be determined by the critical points of the Dirac action (2), see also our corresponding remark in Section 1 on $\mathcal{D}(\xi_{\rm D})$.

Note that it is $\mathrm{End}_{\mathrm{Cl}}(\xi_{\mathrm{D}})$ of the global decomposition of $\mathrm{End}(\xi_{\mathrm{D}})$ that renders the operator ∂ meaningless but, instead, privileges the operators ∂_{A} on general Dirac bundles ξ_{D}.

Proposition 3.1. *For any Dirac bundle $\xi_{\mathrm{D}} = (\mathcal{E}, M, \pi_{\mathcal{E}}, \gamma)$ over an orientable (semi-)Riemannian manifold (M, g_{M}) the Dirac action reads:*

$$\mathcal{S}_{\mathrm{D}}(D) = \int_{\mathrm{M}} *\mathrm{tr}_{\mathcal{E}}\gamma(\mathcal{F}_{\mathrm{D}}) \pm \int_{\partial M} \iota_{X_{\mathrm{D}}}\mu_{\mathrm{M}}. \tag{54}$$

Here, $\mathcal{F}_{\mathrm{D}} \equiv \mathrm{F}_{\nabla_{\mathrm{D}}}$ denotes the curvature with respect to the Dirac connection ∇_{D}, μ_{M} is the (semi-)Riemannian volume form on M, and $\iota_{X_{\mathrm{D}}}$ is the inner derivative (contraction) with respect to the "Dirac (vector) field"

$$X_{\mathrm{D}} := (\mathrm{tr}_{\mathcal{E}}\omega_{\mathrm{D}})^{\sharp} \in \Gamma(\tau_{\mathrm{M}}). \tag{55}$$

Therefore, for closed compact manifolds (i.e. "up to boundary terms") the Dirac action

$$\mathcal{S}_{\mathrm{D}}(D) = \int_{\mathrm{M}} *\mathrm{tr}_{\mathcal{E}}\gamma(\mathcal{F}_{\mathrm{D}}) \tag{56}$$

naturally generalizes the Einstein–Hilbert action (48). Especially, for $\xi_{\mathrm{D}} := \tau_{\Lambda\mathrm{M}}$ one obtains $\mathcal{S}_{\mathrm{D}}(\not{\partial}) \sim \mathcal{S}_{\mathrm{EH}}(g_{\mathrm{M}})$.

Due to the linear isomorphism between the Clifford and the Grassmann algebra, we may identify $\gamma(\mathcal{F}_{\mathrm{D}}) \in \Gamma(\mathrm{End}(\xi_{\mathrm{D}}))$ with $\mathcal{F}_{\mathrm{D}} \in \Omega^2(M, \mathrm{End}(\mathcal{E}))$ and call $\mathcal{F}_{\mathrm{D}} \in \Gamma(\xi^*_{\mathrm{D}} \otimes_{\mathrm{M}} \xi_{\mathrm{D}})$ the *"Dirac curvature"*.

Proof. The statement is an immediate consequence of Theorem 2.1. From the latter it follows that the zero order part associated with (the square of) any Dirac type first order differential operator D actually reads

$$V = \mathcal{F}_{\mathrm{D}} \pm \mathrm{div}_{\mathrm{D}}\omega_{\mathrm{D}}. \tag{57}$$

The assertion then follows from the identity

$$\mathrm{tr}_{\mathcal{E}}(\mathrm{div}_{\mathrm{D}}\omega_{\mathrm{D}}) \equiv \mathrm{div}X_{\mathrm{D}} \tag{58}$$

where "div" refers to the Levi-Civitá connection with respect to g_{M}. Thus,

$$*\mathrm{tr}_{\mathcal{E}}(\mathrm{div}_{\mathrm{D}}\omega_{\mathrm{D}}) = \pounds_{X_{\mathrm{D}}}\,\mu_{\mathrm{M}} \tag{59}$$

with $\pounds_{X_{\mathrm{D}}}$ being the Lie derivative with respect to the Dirac field X_{D} on M. □

The condition (19) thus implies that the (local) flow generated by the Dirac vector field $X_{\mathrm{D}} \in \Gamma(\tau_{\mathrm{M}})$ is volume preserving. Accordingly, we call $D \in \mathcal{D}(\xi_{\mathrm{D}})$ *"unimodular"* if $\mathrm{div}X_{\mathrm{D}} = 0$. Clearly, this condition is weaker than (19). In any case, the two top forms $*\mathrm{tr}_{\mathcal{E}}\mathcal{F}_{\mathrm{D}}$ and $*\mathrm{tr}_{\mathcal{E}}V$ on M are shown to yield the same cohomology class in $\mathrm{H}^{2\mathrm{n}}_{\mathrm{dR}}(M)$. Of course, for this the orientability of M is not really necessary. On the other hand, it is the form (54) of the canonical functional (2) which clearly demonstrates that the Dirac functional is a natural generalization of the Einstein–Hilbert functional.[2]

In [22] we introduced a specific class of gauge theories, called *"gauge theories of Dirac type"*, which permits re-writing the full action functional of the (minimal) Standard Model (including gravity) in terms of an appropriate Dirac type differential operator. In particular, the bosonic action of the Standard Model has been shown to be expressible in terms of the Dirac action (2). Therefore, in the light of the results presented, we put forward the following

Theorem 3.1. *There are (hermitian) Dirac bundles $\xi_{\mathrm{D}} := \xi_{\mathrm{F}}$, called "fermion bundles", and a certain class of Dirac type first order differential operators $D \in \mathcal{D}(\xi_{\mathrm{D}})$, called of "Pauli–Yukawa type", such that (up to boundary terms)*

$$\begin{aligned}\mathcal{S}_{\mathrm{EHYMH}}(g_{\mathrm{M}}, A, \varphi) &:= \lambda_{\mathrm{EH}}\int_{\mathrm{M}} *r_{\mathrm{M}} \pm \lambda_{\mathrm{YM}}\int_{\mathrm{M}} *\kappa(F_{\mathrm{A}}, F_{\mathrm{A}}) \pm \lambda_{\mathrm{H}}\int_{\mathrm{M}} [\partial_{\mathrm{A}}\varphi \wedge *\partial_{\mathrm{A}}\varphi \pm *V_{\mathrm{H}}(\varphi)]\\ &= \lambda_{\mathrm{D}}\int_{\mathrm{M}} *\mathrm{tr}_{\mathcal{E}}\mathcal{F}_{\mathrm{D}},\end{aligned} \tag{60}$$

[2] The author thanks the referee for appropriate comments on this point.

where

$$V_{\rm H}(\varphi) := \lambda\|\varphi\|^4 - \mu^2\|\varphi\|^2 \tag{61}$$

is the well-known Higgs potential and $\lambda_{\rm EH}, \ldots, \lambda_{\rm D}, \lambda, \mu \in \mathbb{R}$ *are appropriate constants. The relative signs depend on the signature and the definition of the Clifford action. Moreover,* $\varphi \in \Gamma(\xi_{\rm H})$ *is a section of a specific sub-vector bundle* $\xi_{\rm H} \subset \xi_{\rm D}$ *(the "Higgs bundle") and* κ *denotes an appropriate inner product on* $\Lambda M \otimes_{\rm M} {\rm Ad}(G)$*, with* ${\rm Ad}(G)$ *being the total space of the adjoint bundle* $\mathfrak{Ad}(G)$ *associated with a chosen principal* G*-bundle.*

Proof. The statement results from the above Proposition 3.1 together with Proposition 5.1 in [22] and the uniqueness of V. Note that the corresponding Dirac type operators are immediate generalizations of those satisfying the condition (17) and are discussed in some detail in Section 5.1 in loc. sit. □

When seen as a specific Dirac type gauge theory, the *total action* of the (minimal) Standard Model takes the simple geometrical form (again, when appropriate boundary conditions are taken into account)

$$\begin{aligned}\mathcal{S}_{\rm SM} &= \int_{\rm M} *[\langle\psi, D\psi\rangle_{\mathcal{E}} + \lambda_{\rm D}{\rm tr}_{\mathcal{E}}\mathcal{F}_{\rm D}] \\ &\equiv \int_{\rm M} \mathcal{L}_{\rm D,tot}(D,\psi),\end{aligned} \tag{62}$$

where $\langle\cdot,\cdot\rangle_{\mathcal{E}}$ is a hermitian product on $\xi_{\rm F}$, $\psi \in \Gamma(\xi_{\rm F})$ and $D = D_{\rm P} \in \mathcal{D}(\xi_{\rm F})$ is of *"Pauli–Yukawa type"*.

For the convenience of the reader, we call in mind that a *"fermion bundle"* $\xi_{\rm F}$ is defined as a specific Dirac bundle whose structure group can be reduced to the sub-group ${\rm Spin}(2n) \times \rho_{\rm F}(G)$ according to the global decomposition (50). The Lie group G is assumed to be semi-simple, compact and real. Likewise, the *"fermion representation"* $\rho_{\rm F}$ of G is assumed to be unitary. We emphasize that both $\psi \in \Gamma(\xi_{\rm F})$ and the Dirac curvature $\mathcal{F}_{\rm D} \in \Gamma(\xi_{\rm F}^* \otimes_{\rm M} \xi_{\rm F})$ carry the same representation $\rho_{\rm F}$.

The full bosonic part of the action of the Standard Model (including Einstein's theory of gravity) can thus be geometrically expressed *linearly in the curvature* of the Dirac type operator which also yields the hermitian form defining the fermionic action on an appropriate fermion bundle.

According to the definition of Dirac type gauge theories, one infers from the results presented that the corresponding top forms of these gauge theories (basically defined by the choice of $D \in \mathcal{D}(\xi_{\rm F})$) have the *universal geometrical* form given by the right-hand side of (62). We mention that the *total Dirac Lagrangian*

$$\begin{aligned}&\mathcal{L}_{\rm D,tot} : \mathcal{D}(\xi_{\rm D}) \times \Gamma(\xi_{\rm D}) \longrightarrow \Omega^{2n}(M) \\ &(D,\psi) \mapsto \mathcal{L}_{\rm D,tot}(D,\psi)\end{aligned} \tag{63}$$

is *equivariant* ("covariant") with respect to the action of the diffeomorphism group of the underlying Dirac bundle, which in the case of Dirac type gauge theories (where $\xi_{\rm D} \equiv \xi_{\rm F}$) decomposes into the semi-direct product

$$\mathcal{G}_{\rm F} \equiv {\rm Diff}(\xi_{\rm F}) = \mathcal{G}_{\rm ex} \ltimes \mathcal{G}_{\rm in} \tag{64}$$

with

$$\begin{aligned}&\mathcal{G}_{\rm ex} \simeq {\rm Diff}(M), \\ &\mathcal{G}_{\rm in} \simeq \mathcal{G}_{\rm EH} \times_{\rm M} \mathcal{G}_{\rm YM}.\end{aligned} \tag{65}$$

Here, $\mathcal{G}_{\rm EH}$ is the gauge group of the SO(p,q)–reduced frame bundle of M corresponding to a fixed Clifford action γ, and $\mathcal{G}_{\rm YM}$ is the appropriate Yang–Mills gauge group considered (cf. Section 2.2. in [22]). A point-wise look at the Einstein–Hilbert gauge group corresponds to the "rotation of SO(p,q)-frames", and the Yang–Mills gauge group can be identified with $\rho_{\rm F}(G)$. The semi-direct decomposition (64) refers to the non-trivial action of the diffeomorphism group of M on $\mathcal{E}$. Indeed, for $f \in {\rm Diff}(M)$ the Clifford action with respect to ${f^{-1}}^* g_{\rm M}$ coincides with the original Clifford action on $\mathcal{E}$ iff f is an isometry.

Therefore, the *"total Dirac action"*

$$\mathcal{S}_{\rm D,tot}(D,\psi) := \int_{\rm M} \mathcal{L}_{\rm D,tot}(D,\psi) \tag{66}$$

is invariant with respect to the action of ${\rm Diff}(\xi_{\rm D})$. Consequently, it formally descends to the quotient set $\mathfrak{M}_{\rm D}$.

The gauge group (64) naturally generalizes the gauge group of Einstein's theory of gravity which, when written in the form (46), reads $\mathrm{Diff}(\tau_{\Lambda M}) \simeq \mathrm{Diff}(M) \ltimes \mathcal{G}_{EH}$. We recall that only the gauge group $\mathcal{G}_{EH}$ can be physically realized. In contrast, the "symmetry" with respect to $\mathrm{Diff}(M)$ reflects the independence of physics on the choice of a particular manifold (background) structure represented by M. However, one may ask how to experimentally select a specific diffeomorphism class $[M]$. This is similar to how distinguish a specific $\mathcal{G}_{YM}$.

The possibility to geometrically consider the Standard Model as a specific gauge theory of Dirac type (and hence to re-write its action in the form (62)) also has predictive power. Indeed, the physical consequences, for example, with respect to the prediction of the mass of the Higgs boson are discussed in detail in [23].

Of course, also the pure Yang–Mills action

$$\mathcal{S}_{YM}(A) := \lambda_{YM} \int_M *\kappa(F_A, F_A) \tag{67}$$

can be expressed *linearly* in the curvature of an appropriate Dirac connection (9) analogous to the Einstein–Hilbert functional. For this one makes use of the fact that also the Yang–Mills curvature can be regarded as a curvature on (the total space of) an appropriate Dirac bundle (cf. again [22]):

$$\mathcal{S}_{YM}(A) \sim \int_M *\mathrm{tr}_{\mathcal{E}}\gamma(F_A^{\mathcal{E}/S})^2. \tag{68}$$

However, for corresponding $D \in \mathcal{D}(\xi_F)$ one obtains

$$\mathcal{S}_{YM}(A) \sim \mathcal{S}_D(D) \tag{69}$$

only if (M, g_M) is assumed to be flat. This reflects that the critical points of the pure Yang–Mills functional are related to the data determining the twisting part of ξ_F only (i.e. to the second factor of the global decomposition (50)). In general, only the combination

$$\mathcal{S}_D(D) \sim \mathcal{S}_{EH}(g_M) + \mathcal{S}_{YM}(A), \tag{70}$$

is natural within Dirac type gauge theories whereby the critical points of this Dirac action yield a mutually dependence of the Clifford action γ and the Yang–Mills connection A. Interestingly, a similar statement holds true for the Higgs functional. Indeed, for appropriate $D \in \mathcal{D}(\xi_F)$ the Higgs functional can be re-written in terms of a Dirac functional

$$\mathcal{S}_H(\varphi) \sim \mathcal{S}_D(D) \tag{71}$$

only if the Yang–Mills connection A (and g_M) is assumed to be flat. Otherwise, one ends up with (60). Therefore, when seen as a particular Dirac type gauge theory, the Higgs action (including the Higgs potential) naturally comes with the (Einstein–Hilbert) Yang–Mills functional.

General remark on the functional $\mathcal{S}_D$: Some remarks may be worth mentioning to avoid confusions with respect to the geometrical form (54) of the Dirac functional. In fact, the reader may wonder how an action linear in the (Dirac) curvature can yield reasonable field equations. The reason is the same as for the Einstein–Hilbert action (46). One has to take the variation not with respect to the (Dirac) connection but with respect to the fields defining this connection. In this sense, the role of Dirac connections (30) with respect to the Dirac action (56) is similar to metric connections in Einstein's theory of gravity.

To make this more precise, let again $\xi_D \simeq \tau^{\mathbb{C}}_{\Lambda M} \otimes_M \xi_E$ be (equivalent to) a twisted Grassmann bundle and

$$\zeta_D := \xi_{EH} \times_M \mathrm{End}(\xi_D) \tag{72}$$

be the fiber bundle over M with total space $\mathcal{Z}$. Also, let $J_1\zeta_D \equiv (J_1\mathcal{Z}, M, \pi_1)$ be the corresponding first jet bundle. A Dirac type first order differential operator $D \in \mathcal{D}(\xi_D)$ may be identified with (an equivalence class of) sections $\sigma \in \Gamma(\zeta_D)$ so that the universal Dirac Lagrangian (12) gives rise to the natural Lagrangian mapping

$$\begin{aligned} \mathcal{L} : J_1\mathcal{Z} &\longrightarrow \Lambda^{2n}M \\ j_1\sigma(x) &\mapsto \mathcal{L}_D(D)|_{x\in M}. \end{aligned} \tag{73}$$

It turns out that this mapping is actually independent of the representative σ and thus well defined (i.e. it only depends on D). Consequently, the Dirac action may be considered equally well as a functional of $\sigma = (g_M, \Phi)$ with

$\Phi \in \Gamma(\mathrm{End}(\xi_D))$ being a *"(bosonic) super-field"* taking values in $\mathrm{End}_{Cl}(\xi_D)$. Likewise, every choice of $D \in \mathcal{D}(\xi_D)$ fixes a super-field Φ corresponding to D (and hence to an appropriate Dirac connection). When expressed in terms of the "components" of such a super-field, the universal Dirac action gives rise to a specific functional of these components (and the metric) like in the case (60). Note that this is quite similar to super symmetric actions.

4. Conclusion, remarks and outlook

We presented a globally geometrical formula for the zero order operator uniquely associated with every Dirac type first order differential operator. The form of this general Lichnerowicz formula (7) is as similar as possible to the well-known Lichnerowicz formula (4) for Dirac type operators defined in terms of Clifford connections. We also presented some natural restrictions on the domain of the canonical functional (2), see also below. A further study of these restrictions will certainly be useful to gain some geometrical understanding of the moduli space $\mathfrak{M}_D$ of Dirac type first order differential operators.

Furthermore, we proved the Dirac functional to be a natural generalization of the Einstein–Hilbert action of general relativity. In fact, it follows that the full bosonic action of the Standard Model can be re-written linearly in the curvature of an appropriate ("Pauli–Yukawa" type) Dirac operator in complete analogy with the Einstein–Hilbert action. In particular, this holds true also for the pure Yang–Mills theory.

The study of the structure of $\mathfrak{M}_D$ may turn out to be useful to also obtain a deeper geometrical understanding of the action on which the Standard Model is based. Furthermore, such an analysis is also needed to systematically go beyond the Standard Model within the geometrical frame of Dirac type gauge theories. For this it will be interesting to also discuss the (total) Dirac action from the point of view of (super-symmetric) Morse theory and how the critical points of $\mathcal{S}_D(D)$ are related to the *"generalized Yang–Mills equations"* naturally associated with every Dirac type first order differential operator $D \in \mathcal{D}(\xi_D)$:

$$d_D\mathcal{F}_D = \delta_D\mathcal{F}_D = 0, \tag{74}$$

where d_D is the exterior covariant derivative with respect to the Dirac connection ∇_D and δ_D its formal adjoint. Clearly, these equations are but the Euler–Lagrange equations of the *generalized Yang–Mills action*

$$\mathcal{S}_{DYM}(D) := \lambda_{DYM} \int_M \mathrm{tr}_{\mathcal{E}}(\mathcal{F}_D \wedge *\mathcal{F}_D), \tag{75}$$

considered as a functional of (Dirac) connections on ξ_D. Note that, when expressed in terms of the components of the super-field defining D, the action (75) generically yields fourth order derivatives of g_M. This is one reason why gravity in "Yang–Mills-like form" causes difficulties both classically and when one tries to quantize it. This, however, does not hold true as long as the action (75) is treated as a functional of connections on ξ_D as in ordinary Yang–Mills gauge theory (please, see also the corresponding remarks below).

The generalized Yang–Mills equation (74) may be re-written as

$$D\mathcal{F}_D = 0 \tag{76}$$

according to the identification $D = \gamma(\nabla_D) = d_D + \delta_D$, which generalizes the Gauss–Bonnet operator (45) to arbitrary Dirac bundles ξ_D. Especially, the condition (76) on $D \in \mathcal{D}(\xi_D)$ clearly generalizes the usual Yang–Mills equations formulated on general fermion bundles ξ_F with the metric g_M being *arbitrary* but *fixed*:

$$\partial\!\!\!/_A\, F_A^{\mathcal{E}/S} = 0. \tag{77}$$

For various principal G-bundles over certain (M, g_M) and fermion bundles of the specific form $\xi_F := \tau^{\mathbb{C}}_{\Lambda M} \otimes_M \mathfrak{Ad}(G)$ a similar form of the Yang–Mills equations is extensively studied from the point of view of Clifford analysis (see, for example, [9,10]). Indeed, the Maxwell equations on Minkowski space–time (whereby the corresponding Grassmann bundle can be naturally regarded as a specific fermion bundle) have been introduced in a form similar to (77) by various authors within the description of electrodynamics in terms of Clifford's geometric algebra (see, for example, in [12,14,13,17,19]; see also in [11,16] and the corresponding references therein, in particular, for historical remarks concerning Maxwell's equations and Clifford algebra).

Note that for the Gauss–Bonnet operator $\Gamma(\tau_{\Lambda M}) \stackrel{\not\partial}{\longrightarrow} \Gamma(\tau_{\Lambda M})$, the generalized Yang–Mills equation (76) reduces to the *"homogeneous gravitational equation"*:

$$\not\partial \mathcal{R} = 0. \tag{78}$$

Of course, this equation can be easily put forward to general Dirac bundles by the replacement (47):

$$\not\partial_{\mathrm{A}} \mathcal{R} = 0, \tag{79}$$

where now the twisting (Yang–Mills) part A is considered *arbitrary* but *fixed*.

We stress that this "covariant" (homogeneous) gravitational equation is well defined on general Dirac bundles ξ_{D}. This is mainly due to the decomposition (50). As a consequence, the total curvature F_{A} of a Clifford connection $\partial_{\mathrm{A}} \in \mathcal{A}(\xi_{\mathrm{D}})$ globally decomposes into

$$\begin{aligned} \mathrm{F}_{\mathrm{A}} &= \mathcal{R} \otimes \mathrm{id} + \mathrm{id} \otimes F_{\mathrm{A}}^{\mathcal{E}/S} \\ &\equiv \mathcal{R} + F_{\mathrm{A}}^{\mathcal{E}/S}. \end{aligned} \tag{80}$$

Hence, (79) locally coincides with (78) and together with the Yang–Mills equation (77) corresponds to the single equation[3]

$$\not\partial_{\mathrm{A}} \mathrm{F}_{\mathrm{A}} = 0. \tag{81}$$

However, this field equation obviously does not give rise to physically admissible field equations neither for the gravitational field, nor for the Yang–Mills gauge field. Indeed, (81) does *not* correspond to (76). This is because the Dirac action with respect to $\not\partial_{\mathrm{A}}$ is independent of the twisting part of ∂_{A}, as mentioned already. It therefore only yields the *homogeneous* field equation of gravity, see (49) and the remarks thereafter. Likewise, the metric used in (68) to derive the Yang–Mills equation (77) is arbitrarily chosen. To remedy this flaw of (81) one has to consider more general Dirac connections than those satisfying the condition (16). In this respect it will be worthwhile to further investigate the class of *"Pauli–Dirac type"* first order differential operators introduced in [22] (please, see Def. 5.1 in [22]).

In the case of $\dim(M) = 4$, the relation between the critical points of the Dirac action (2) and the (anti-)self-dual solutions

$$*\mathcal{F}_{\mathrm{D}} = \pm \mathcal{F}_{\mathrm{D}} \tag{82}$$

of (74) (resp. of (76)) are of particular interest. This will be discussed in more detail in a forthcoming work with the intention to study the universal Dirac functional more closely and to further classify Dirac type first order differential operators.

For a description of Einstein's theory of gravity in terms of Clifford algebras see, for example, [20]. In fact, from the statements presented there, one may infer that there is a deep link between the critical points of the generalized Yang–Mills action (75) and the Dirac action (2). In this sense the former functional may be considered as the "square" of the latter functional formally analogous to the general Lichnerowicz decomposition (1) we started out our discussion.

Acknowledgments

The author would like to thank E. Binz and E. Zeidler for their continuous interest and stimulating discussions on the presented subject.

References

[1] T. Ackermann, J. Tolksdorf, The generalized Lichnerowicz formula and analysis of Dirac operators, J. Reine Angew. Math. (1996).

[3] Unfortunately, in the case of an *abelian* Yang–Mills gauge group $\mathcal{G}_{\mathrm{YM}} \subset \mathcal{G}_{\mathrm{F}}$, like in electromagnetism, there is a confusingly coincidence between the Eq. (81) and $\not\partial \mathrm{F}_{\mathrm{A}} = 0$ (i.e., $\not\partial \mathcal{R} = \not\partial F_{\mathrm{A}}^{\mathcal{E}/S} = 0$). The latter equations are well defined, actually, although the Gauss–Bonnet operator $\not\partial$ has no global meaning even in the case of a fermion bundle ξ_{F} with abelian sub-group $\mathcal{G}_{\mathrm{YM}}$.

[2] M.F. Atiyah, R. Bott, A. Shapiro, Clifford modules, Topology 3 (1964) 3–38.
[3] N. Berline, E. Getzler, M. Vergne, Heat Kernels and Dirac Operators, Springer-Verlag, 1996.
[4] J.M. Bismut, The Atiyah–Singer index theorem for families of Dirac operators: Two heat equation proofs, in: Mathematicae, Springer-Verlag, 1986.
[5] T. Branson, P.B. Gilkey, The asymptotics of the Laplacian on a manifold with boundary, Comm. Partial Differential Equations 15 (1990) 245–272.
[6] A. Connes, Gravity coupled with matter and foundation of noncommutative geometry, Comm. Math. Phys. 182 (1996) 155–176.
[7] P. Gilkey, Invariance Theory, the Heat Equation, and the Atiyah–Singer Index Theorem, in: Studies in Advanced Mathematics, CRC Press, 1995.
[8] W. Greub, Multilinear Algebra, 2nd edn, Springer-Verlag, New York, 1978.
[9] F. Gürsey, H.C. Tze, Complex and quaternionic analyticity in chiral and gauge theories I, Ann. Phys. 128, 29–130.
[10] F. Gürsey, H.C. Tze, On the Role of Division, Jordan and Related Algebras in Particle Physics, World Scientific, Singapore, 1996.
[11] D. Hestenes, Space–Time Algebra, Gordon and Breach, New York, 1992.
[12] G. Juvet, Opérateurs de Dirac et équations de Maxwell, Comment. Math. Helv. 2 (1930) 225–235.
[13] G. Juvet, A. Schidlof, Sur les nombres hypercomplexes de Clifford et leurs applications à l'analyse vectorielle ordinaire, à l'électromagnesime de Minkowski et à la théorie de Dirac, Bull. Soc. Neuchat. Sci. Nat. 57 (1932) 127–147.
[14] O. Laporte, G.E. Uhlenbeck, Application of spinor analysis to the Maxwell and Dirac equations, Phys. Rev. 37 (1931) 1380–1397.
[15] A. Lichnerowicz, Spineurs harmonique, C. R. Acad. Sci. Paris, Ser. A 257 (1963) 7–9.
[16] P. Lounesto, Clifford Algebras and Spinors, in: London Mathematical Society Lecture Note Series, vol. 239, Cambridge Univ. Press, 1997.
[17] A. Mercier, Expression des équations de l'éctromagnétisme au moyen des nombres de Clifford, Thesis, Université de Genève, 1935.
[18] D. Quillen, Superconnections and the Chern character, Topology 24 (1) (1985) 89–95.
[19] M. Riesz, Clifford Numbers and Spinors, in: The Inst. for Fluid Dynamics and Applied Mathematics, in: Lecture Series No., vol. 38, University of Maryland, 1958, Reprinted by E.F. Bolinder, P. Lounesto (Eds), Kluwer, Dordrecht, The Netherlands, 1993.
[20] E. Stedile, Clifford algebra structure and Einstein's equations, Internat. J. Theoret. Phys. 35 (10) (1996).
[21] J. Tolksdorf, The Einstein–Hilbert–Yang–Mills–Higgs action and the Dirac–Yukawa operator, J. Math. Phys. 39 (1998).
[22] J. Tolksdorf, T. Thumstädter, Gauge theories of Dirac type, J. Math. Phys. 47 (2006) 082305.
[23] J. Tolksdorf, T. Thumstädter, Dirac type gauge theories and the mass of the Higgs, J. Phys. A: Math. Theoret. (submitted for publication), math-ph/0602028.